

ON PROPERTIES OF SOLUTIONS OF THE p -HARMONIC EQUATION

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ABSTRACT. A $2p$ -times continuously differentiable complex-valued function $f = u + iv$ in a simply connected domain $\Omega \subseteq \mathbb{C}$ is p -harmonic if f satisfies the p -harmonic equation $\Delta^p f = 0$. In this paper, we investigate the properties of p -harmonic mappings in the unit disk $|z| < 1$. First, we discuss the convexity, the starlikeness and the region of variability of some classes of p -harmonic mappings. Then we prove the existence of Landau constant for the class of functions of the form $Df = zf_z - \bar{z}f_{\bar{z}}$, where f is p -harmonic in $|z| < 1$. Also, we discuss the region of variability for certain p -harmonic mappings. At the end, as a consequence of the earlier results of the authors, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings.

1. INTRODUCTION AND PRELIMINARIES

A complex-valued function $f = u + iv$ in a simply connected domain $\Omega \subseteq \mathbb{C}$ is called p -harmonic if u and v are p -harmonic in Ω , i.e. f satisfies the p -harmonic equation $\Delta^p f = 0$, where

$$\Delta^p f = \underbrace{\Delta \cdots \Delta}_p f,$$

where p is a positive integer and Δ represents the Laplacian operator

$$\Delta := 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Throughout this paper we consider p -harmonic mappings of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Obviously, when $p = 1$ (resp. $p = 2$), f is harmonic (resp. biharmonic). The properties of harmonic [11, 15] and biharmonic [1, 2, 3, 18, 19] mappings have been investigated by many authors. Concerning p -harmonic mappings, we easily have the following characterization.

Proposition 1. *A mapping f is p -harmonic in \mathbb{D} if and only if f has the following representation:*

$$(1.1) \quad f(z) = \sum_{k=1}^p |z|^{2(k-1)} G_{p-k+1}(z),$$

where G_{p-k+1} is harmonic for each $k \in \{1, \dots, p\}$.

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Proof. We only need to prove the necessity since the proof for the sufficiency part is obvious. Again, as the cases $p = 1, 2$ are well-known, it suffices to prove the result for $p \geq 3$. We shall prove the proposition by the method of induction. So, we assume that the proposition is true for $p = n (\geq 3)$.

Let F be an $(n+1)$ -harmonic mapping in \mathbb{D} . By assumption, ΔF is n -harmonic and so can be represented as

$$\Delta F(z) = \sum_{k=1}^n |z|^{2(k-1)} G_{n-k+1}(z),$$

where G_{n-k+1} ($1 \leq k \leq n$) are harmonic functions with

$$G_{n-k+1}(z) = a_{0,n-k+1} + \sum_{j=1}^{\infty} a_{j,n-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,n-k+1} \bar{z}^j \quad \text{for } k \in \{1, \dots, n\}.$$

Then

$$\int_0^z \int_0^{\bar{z}} \Delta F \, d\bar{z} \, dz = \sum_{k=1}^n |z|^{2k} T_{p-k+1}(z) + g(z),$$

where

$$T_{p-k+1}(z) = \sum_{k=1}^n \left(\frac{a_{0,n-k+1}}{k^2} + \sum_{j=1}^{\infty} \frac{a_{j,n-k+1}}{k(k+j)} z^j + \sum_{j=1}^{\infty} \frac{\bar{b}_{j,n-k+1}}{k(k+j)} \bar{z}^j \right)$$

and g is a harmonic function in \mathbb{D} . A rearrangement of the series in the sum shows that (1.1) holds for $p = n+1$. \square

We remark that the representation (1.1) continues to hold even if f is p -harmonic in a simply connected domain Ω .

For a sense-preserving C^1 -mapping (i.e. continuously differentiable), we let

$$\lambda_f = |f_z| - |f_{\bar{z}}| \quad \text{and} \quad \Lambda_f = |f_z| + |f_{\bar{z}}|$$

so that the Jacobian J_f of f takes the form

$$J_f = \lambda_f \Lambda_f = |f_z|^2 - |f_{\bar{z}}|^2 > 0.$$

In [4], the authors obtained sufficient conditions for the univalence of C^1 -functions. Now we introduce the concepts of starlikeness and convexity of C^1 -functions.

Definition 1. A C^1 -mapping f with $f(0) = 0$ is called starlike if f maps \mathbb{D} univalently onto a domain Ω that is starlike with respect to the origin, i.e. for every $w \in \Omega$ the line segment $[0, w]$ joining 0 and w is contained in Ω . It is known that f is starlike if it is sense-preserving, $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and

$$\frac{\partial}{\partial t} (\arg f(re^{it})) := \operatorname{Re} \left(\frac{Df(z)}{f(z)} \right) > 0 \quad \text{for all } z = re^{it} \in \mathbb{D} \setminus \{0\},$$

where $Df = zf_z - \bar{z}f_{\bar{z}}$ (cf. [23, Theorem 1]).

Definition 2. Let f and Df belong to $C^1(\mathbb{D})$. Then we say that f is *convex* in \mathbb{D} if it is sense-preserving, $f(0) = 0$, $f(z) \cdot Df(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and

$$\operatorname{Re} \left(\frac{D^2 f(z)}{Df(z)} \right) > 0 \quad \text{for all } z \in \mathbb{D} \setminus \{0\}.$$

As $\arg Df(re^{it})$ represents the argument of the outer normal to the curve $C_r = \{f(re^{i\theta}) : 0 \leq \theta < 2\pi\}$ at the point $f(re^{it})$, the last condition gives that

$$\frac{\partial}{\partial t} (\arg Df(re^{it})) = \operatorname{Re} \left(\frac{D^2 f(z)}{Df(z)} \right) > 0 \quad \text{for all } z = re^{it} \in \mathbb{D} \setminus \{0\},$$

showing that the curve C_r is convex for each $r \in (0, 1)$ (see [23, Theorem 2]). Non-analytic starlike and convex functions were studied by Mocanu in [23]. Harmonic starlike and harmonic convex functions were systematically studied by Clunie and Sheil-Small [11], and these two classes of functions have been studied extensively by many authors. See for instance, the book by Duren [15] and the references therein.

The complex differential operator

$$D = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}$$

defined by Mocanu [23] on the class of complex-valued C^1 -functions satisfies the usual product rule:

$$D(af + bg) = aD(f) + bD(g) \quad \text{and} \quad D(fg) = fD(g) + gD(f),$$

where a, b are complex constants, f and g are C^1 -functions. The operator D possesses a number of interesting properties. For instance, the operator D preserves both harmonicity and biharmonicity (see also [3]). In the case of p -harmonic mappings, we also have the following property of the operator D .

Proposition 2. D preserves p -harmonicity.

Proof. Let f be a p -harmonic mapping with the form

$$f(z) = \sum_{k=1}^p |z|^{2(k-1)} G_{p-k+1}(z),$$

where each $G_{p-k+1}(z)$ is harmonic in \mathbb{D} for $k \in \{1, \dots, p\}$. As $D(|z|^2) = 0$, the product rule shows that $D(|z|^{2(k-1)}) = 0$ for each $k \in \{1, \dots, p\}$. In view of this and the fact that D preserves harmonicity gives that

$$\begin{aligned} D(f(z)) &= \sum_{k=1}^p \left[|z|^{2(k-1)} D(G_{p-k+1}(z)) + D(|z|^{2(k-1)}) G_{p-k+1}(z) \right] \\ &= \sum_{k=1}^p |z|^{2(k-1)} D(G_{p-k+1}(z)). \end{aligned}$$

□

One of the aims of this paper is to generalize the main results of Abdulhadi, et. al. [3] to the case of p -harmonic mappings. The corresponding generalizations are Theorems 1 and 2.

The classical theorem of Landau for bounded analytic functions states that if f is analytic in \mathbb{D} with $f(0) = f'(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent in the disk $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ and in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$ (cf. [20]), where

$$\rho = \frac{1}{M + \sqrt{M^2 - 1}}.$$

Recently, many authors considered Landau's theorem for planar harmonic mappings (see for example, [6, 8, 9, 13, 16, 22, 28]) and biharmonic mappings (see [1, 7, 8, 21]). In Section 4, we consider Landau's theorem for p -harmonic mappings with the form $D(f)$ when f belongs to certain classes of p -harmonic mappings. Our results are Theorems 3 and 4.

In a series of papers the second author with Yanagihara and Vasudevarao (see [24, 25, 29, 30]) have discussed the regions of variability for certain classes of univalent analytic functions in \mathbb{D} . In Section 5 (see Theorem 5), we solve a related problem for certain p -harmonic mappings. Finally, in Section 6, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings (see Corollaries 3 and 4).

2. LEMMAS

For the proofs of our main results we require a number of lemmas. We begin to recall the following version of Schwarz lemma due to Heinz ([17, Lemma]) and Colonna [12, Theorem 3], see also [6, 8, 9].

Lemma A. *Let f be a harmonic mapping of \mathbb{D} such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi} |z| \text{ for } z \in \mathbb{D}$$

and

$$\Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{(1 - |z|^2)} \text{ for } z \in \mathbb{D}.$$

Lemma B. ([22, Lemma 2.1]) *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of \mathbb{D} with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in \mathbb{D}$. If $J_f(0) = 1$ and $|f(z)| < M$, then*

$$|a_n|, |b_n| \leq \sqrt{M^2 - 1}, \quad n = 2, 3, \dots,$$

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \quad n = 2, 3, \dots$$

and

$$(2.1) \quad \lambda_f(0) \geq \lambda_0(M) := \begin{cases} \frac{\sqrt{2}}{\sqrt{M^2 - 1} + \sqrt{M^2 + 1}} & \text{if } 1 \leq M \leq M_0, \\ \frac{\pi}{4M} & \text{if } M > M_0, \end{cases}$$

where $M_0 = \frac{\pi}{2\sqrt[4]{2\pi^2 - 16}} \approx 1.1296$.

The following lemma concerning coefficient estimates for harmonic mappings is crucial in the proofs of Theorems 1 and 2. This lemma has been proved by the authors in [10] with an additional assumption that $f(0) = 0$. However, for the sake of clarity, we present a slightly different proof than that in [10].

Lemma C. *Let $f = h + \bar{g}$ be a harmonic mapping of \mathbb{D} such that $|f(z)| < M$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Then $|a_0| \leq M$ and for any $n \geq 1$*

$$(2.2) \quad |a_n| + |b_n| \leq \frac{4M}{\pi}.$$

The estimate (2.2) is sharp. The extremal functions are $f(z) \equiv M$ or

$$f_n(z) = \frac{2M\alpha}{\pi} \arg \left(\frac{1 + \beta z^n}{1 - \beta z^n} \right),$$

where $|\alpha| = |\beta| = 1$.

Proof. Without loss of generality, we assume that $|f(z)| < 1$. For $\theta \in [0, 2\pi)$, let

$$v_\theta(z) = \operatorname{Im} (e^{i\theta} f(z))$$

and observe that

$$v_\theta(z) = \operatorname{Im} (e^{i\theta} h(z) + \overline{e^{-i\theta} g(z)}) = \operatorname{Im} (e^{i\theta} h(z) - e^{-i\theta} g(z)).$$

Because $|v_\theta(z)| < 1$, it follows that

$$e^{i\theta} h(z) - e^{-i\theta} g(z) \prec K(z) = \lambda + \frac{2}{\pi} \log \left(\frac{1 + z\xi}{1 - z} \right),$$

where $\xi = e^{-i\pi \operatorname{Im}(\lambda)}$ and $\lambda = e^{i\theta} h(0) - e^{-i\theta} g(0)$. The superordinate function $K(z)$ maps \mathbb{D} onto a convex domain with $K(0) = \lambda$ and $K'(0) = \frac{2}{\pi}(1 + \xi)$, and therefore, by a theorem of Rogosinski [26, Theorem 2.3] (see also [14, Theorem 6.4]), it follows that

$$|a_n - e^{-2i\theta} b_n| \leq \frac{2}{\pi} |1 + \xi| \leq \frac{4}{\pi} \quad \text{for } n = 1, 2, \dots$$

and the desired inequality (2.2), with $M = 1$, is a consequence of the arbitrariness of θ in $[0, 2\pi)$.

For the proof of sharpness part, consider the functions

$$f_n(z) = \frac{2M\alpha}{\pi} \operatorname{Im} \left(\log \frac{1 + \beta z^n}{1 - \beta z^n} \right), \quad |\alpha| = |\beta| = 1,$$

whose values are confined to a diametral segment of the disk \mathbb{D}_M . Also,

$$f_n(z) = \frac{2M\alpha}{i\pi} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} (\beta z^n)^{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k-1} (\bar{\beta} \bar{z}^n)^{2k-1} \right),$$

which gives

$$|a_n| + |b_n| = \frac{4M}{\pi}.$$

The proof of the lemma is complete. \square

As an immediate consequence of Lemmas B and C, we have

Corollary 1. *Let $f = h + \bar{g}$ be a harmonic mapping of \mathbb{D} with $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $|f(z)| \leq M$. If $J_f(0) = 1$ and $M \geq \frac{\pi}{\sqrt{\pi^2 - 8}}$, then for any $n \geq 2$,*

$$|a_n| + |b_n| \leq \frac{4M}{\pi} \leq \sqrt{2M^2 - 2}.$$

3. THE CONVEXITY AND THE STARLIKENESS

The following simple result can be used to generate (harmonic) starlike and convex functions.

Theorem 1. *Let f be a univalent p -harmonic mapping with the form*

$$f(z) = G(z) \sum_{k=1}^p \lambda_k |z|^{2(k-1)},$$

where G is a locally univalent harmonic mapping and λ_k ($k = 1, \dots, p$) are complex constants. Then we have the following:

- (a) $\frac{D(f)}{f} = \frac{D(G)}{G}$ and $\frac{D^2(f)}{D(f)} = \frac{D^2(G)}{D(G)}$.
- (b) f is convex (resp. starlike) if and only if G is convex (resp. starlike).

Proof. (a) The two equalities are immediate consequences of the formula

$$D\left(G(z) \sum_{k=1}^p \lambda_k |z|^{2(k-1)}\right) = D(G(z)) \sum_{k=1}^p \lambda_k |z|^{2(k-1)}.$$

So, we omit the details.

(b) It suffices to prove the case of convexity since the proof for the starlikeness is similar.

Let $z = re^{it}$, where $0 < r < 1$ and $0 \leq t < 2\pi$. Then

$$f(z) = G(z) \sum_{k=1}^p \lambda_k |z|^{2(k-1)} = G(re^{it}) \sum_{k=1}^p \lambda_k r^{2(k-1)},$$

so that

$$\frac{\partial f(re^{it})}{\partial t} = \frac{\partial G(re^{it})}{\partial t} \sum_{k=1}^p \lambda_k r^{2(k-1)}$$

and

$$\frac{\partial^2 f(re^{it})}{\partial t^2} = \frac{\partial^2 G(re^{it})}{\partial t^2} \sum_{k=1}^p \lambda_k r^{2(k-1)}.$$

Therefore Part (a) yields

$$\frac{\partial}{\partial t} \left(\arg \frac{\partial f(re^{it})}{\partial t} \right) = \operatorname{Re} \left(\frac{D^2(f)}{D(f)} \right) = \operatorname{Re} \left(\frac{D^2(G)}{D(G)} \right) = \frac{\partial}{\partial t} \left(\arg \frac{\partial G(re^{it})}{\partial t} \right),$$

from which the proof of Part (b) of this theorem follows. \square

As an immediate consequence of Theorem 1(a), we easily have the following.

Corollary 2. *Let f be a univalent p -harmonic mapping defined as in Theorem 1. If f is convex and $D(f)$ is univalent, then $D(f)$ is starlike.*

Abdulhadi, et. al. [3, Theorem 1] discussed the univalence and the starlikeness of biharmonic mappings in \mathbb{D} . A natural question is whether [3, Theorem 1] holds for p -harmonic mappings. The following result gives a partial answer to this problem.

Theorem 2. *Let f be a p -harmonic mapping of \mathbb{D} satisfying $f(z) = |z|^{2(p-1)}G(z)$, where G is harmonic, orientation preserving and starlike. Then f is starlike univalent.*

Proof. We see that the Jacobian J_f of f is

$$\begin{aligned} J_f &= |f_z|^2 - |f_{\bar{z}}|^2 \\ &= |z|^{4(p-1)}(|G_z|^2 - |G_{\bar{z}}|^2) + 2(p-1)|z|^{4p-6}|G|^2 \operatorname{Re} \left(\frac{D(G)}{G} \right) \\ &\geq |z|^{4(p-1)}(|G_z|^2 - |G_{\bar{z}}|^2). \end{aligned}$$

Hence $J_f(z) > 0$ when $0 < |z| < 1$ and obviously, $J_f(0) = 0$. The univalence of f follows from a standard argument as in the proof of [3, Theorem 1]. Finally, Theorem 1 implies that f is starlike. \square

4. THE LANDAU THEOREM

We now discuss the existence of the Landau constant for two classes of p -harmonic mappings.

Theorem 3. *Let $f(z) = \sum_{k=1}^p |z|^{2(k-1)}G_{p-k+1}(z)$ be a p -harmonic mapping of \mathbb{D} satisfying $\Delta G_{p-k+1}(z) = f(0) = G_p(0) = J_f(0) - 1 = 0$ and for any $z \in \mathbb{D}$, $|G_{p-k+1}(z)| \leq M$, where $M \geq 1$. Then there is a constant ρ ($0 < \rho < 1$) such that $D(f)$ is univalent in \mathbb{D}_ρ , where ρ satisfies the following equation:*

$$\lambda_0(M) - \frac{T(M)}{(1-\rho)^2} \sum_{k=2}^p (2k-1)\rho^{2(k-1)} - \sum_{k=1}^p \frac{2T(M)\rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho = 0$$

with

$$(4.1) \quad s_0 = \left(\frac{\sqrt{17}-1}{\sqrt{17}-3} \right) \sqrt{\frac{2}{5-\sqrt{17}}} \approx 4.1996,$$

$$T(M) = \begin{cases} \sqrt{2M^2-2} & \text{if } 1 \leq M \leq M_1 := \frac{\pi}{\sqrt{\pi^2-8}} \approx 2.2976 \\ \frac{4M}{\pi} & \text{if } M > M_1 \end{cases}$$

and $\lambda_0(M)$ is given by (2.1). Moreover, the range $D(f)(\mathbb{D}_\rho)$ contains a univalent disk \mathbb{D}_R , where

$$R = \rho \left[\lambda_0(M) - \sum_{k=2}^p \frac{T(M)\rho^{2(k-1)}}{(1-\rho)^2} - \frac{16M}{\pi^2} s_0 \arctan \rho \right].$$

Proof. For each $k \in \{1, 2, \dots, p\}$, let

$$G_{p-k+1}(z) = a_{0,p-k+1} + \sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p-k+1} \bar{z}^j,$$

where $a_{0,p} = 0$. We define the function H as

$$H = D \left(\sum_{k=1}^p |z|^{2(k-1)} G_{p-k+1} \right) = \sum_{k=1}^p |z|^{2(k-1)} D(G_{p-k+1}).$$

Using Lemmas B, C and Corollary 1, we have

$$|a_{n,p}| + |b_{n,p}| \leq T(M),$$

where $T(M)$ is given by (4.1), and

$$|a_{j,p-k+1}| + |b_{j,p-k+1}| \leq \frac{4M}{\pi}$$

for $j \geq 1$, $n \geq 2$ and $2 \leq k \leq p$.

We observe that

$$J_f(0) = |(G_p)_z(0)|^2 - |(G_p)_{\bar{z}}(0)|^2 = J_{G_p}(0) = 1$$

and hence by Lemmas A and B, we have

$$\lambda_f(0) \geq \lambda_0(M),$$

where $\lambda_0(M)$ is given by (2.1). Now, we define

$$q(x) = \frac{2-x^2}{(1-x^2)x} \quad (0 < x < 1).$$

Then there is an $r_0 = \sqrt{\frac{5-\sqrt{17}}{2}} \approx 0.66$ such that

$$q(r_0) = \min_{0 < x < 1} q(x) = \left(\frac{\sqrt{17}-1}{\sqrt{17}-3} \right) \sqrt{\frac{2}{5-\sqrt{17}}} = s_0.$$

For each $\theta \in [0, 2\pi)$, the function

$$G_\theta(z) = (G_p)_z(z) - (G_p)(0) + ((G_p)_{\bar{z}}(z) - (G_p)_{\bar{z}}(0))e^{i(\pi-2\theta)}$$

is clearly a harmonic mapping of \mathbb{D} and satisfies $G_\theta(0) = 0$. Moreover, it follows from Lemma A that

$$\Lambda_{G_p}(z) \leq \frac{4M}{\pi} \frac{1}{1-|z|^2} \quad \text{for } z \in \mathbb{D}.$$

In particular, this observation yields that

$$(4.2) \quad |G_\theta(z)| \leq \Lambda_{G_p}(z) + \Lambda_{G_p}(0) \leq \frac{4M}{\pi} \left(1 + \frac{1}{1-|z|^2}\right) = \frac{4M}{\pi} |z|q(|z|)$$

for all $z \in \mathbb{D}$.

Since $xq(x) - 1 = \frac{1}{1-x^2}$ is an increasing function in the interval $(0, 1)$, the inequality (4.2) shows that for any $z \in \mathbb{D}_{r_0}$,

$$|G_\theta(z)| \leq \frac{4M}{\pi} m_0,$$

where $m_0 = (2 - r_0^2)/(1 - r_0^2)$. Next, we consider the mapping F defined on \mathbb{D} by

$$F(z) = \frac{\pi}{4Mm_0} G_\theta(r_0 z).$$

Applying Lemma A to the function $F(z)$ yields that for $z \in \mathbb{D}_{r_0}$,

$$|G_\theta(z)| \leq \frac{16M}{\pi^2} m_0 \arctan \left(\frac{|z|}{r_0} \right) \leq \frac{16M}{\pi^2} s_0 \arctan |z|,$$

where $s_0 = m_0/r_0$.

Now, we fix ρ with $\rho \in (0, 1)$. To prove the univalence of H , we choose two distinct points z_1, z_2 in \mathbb{D}_ρ . Let $\gamma = \{(z_2 - z_1)t + z_1 : 0 \leq t \leq 1\}$ and $z_2 - z_1 = |z_1 - z_2|e^{i\theta}$. We find that

$$\begin{aligned}
& |H(z_1) - H(z_2)| \\
&= \left| \int_{\gamma} H_z(z) dz + H_{\bar{z}}(z) d\bar{z} \right| \\
&\geq \left| \int_{\gamma} (G_p)_z(0) dz - (G_p)_{\bar{z}}(0) d\bar{z} \right| \\
&\quad - \left| \int_{\gamma} \sum_{k=2}^p |z|^{2(k-1)} [z(G_{p-k+1})_{zz}(z) dz - \bar{z}(G_{p-k+1})_{\bar{z}\bar{z}}(z) d\bar{z}] \right| \\
&\quad - \left| \int_{\gamma} \sum_{k=2}^p (k-1) |z|^{2(k-2)} [z^2(G_{p-k+1})_z(z) d\bar{z} - \bar{z}^2(G_{p-k+1})_{\bar{z}}(z) dz] \right| \\
&\quad - \left| \int_{\gamma} \sum_{k=2}^p k |z|^{2(k-1)} [(G_{p-k+1})_z(z) dz - (G_{p-k+1})_{\bar{z}}(z) d\bar{z}] \right| \\
&\quad - \left| \int_{\gamma} [(G_p)_z(z) - (G_p)_z(0)] dz - [(G_p)_{\bar{z}}(z) - (G_p)_{\bar{z}}(0)] d\bar{z} \right| \\
&\geq |z_1 - z_2| \left\{ \lambda_f(0) - |G_{\theta}(\rho)| \right. \\
&\quad - \sum_{k=1}^p \rho^{2(k-1)} \sum_{n=2}^{\infty} n(n-1) (|a_{n,p-k+1}| + |b_{n,p-k+1}|) \rho^{n-1} \\
&\quad \left. - \sum_{k=2}^p (2k-1) \rho^{2(k-2)} \sum_{n=1}^{\infty} n (|a_{n,p-k+1}| + |b_{n,p-k+1}|) \rho^{n+1} \right\} \\
&> |z_1 - z_2| \left[\lambda_0(M) - \frac{T(M)}{(1-\rho)^2} \sum_{k=2}^p (2k-1) \rho^{2(k-1)} \right. \\
&\quad \left. - \sum_{k=1}^p \frac{2T(M) \rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho \right].
\end{aligned}$$

Let

$$P(\rho) = \lambda_0(M) - \frac{T(M)}{(1-\rho)^2} \sum_{k=2}^p (2k-1) \rho^{2(k-1)} - \sum_{k=1}^p \frac{2T(M) \rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho.$$

Then it is easy to verify that $P(\rho)$ is a decreasing function on the interval $(0, 1)$,

$$\lim_{\rho \rightarrow 0+} P(\rho) = \lambda_0(M) \quad \text{and} \quad \lim_{\rho \rightarrow 1-} P(\rho) = -\infty.$$

Hence there exists a unique ρ_0 in $(0, 1)$ satisfying $P(\rho_0) = 0$. This observation shows that $|H(z_1) - H(z_2)| > 0$ for arbitrary two distinct points z_1, z_2 in $|z| < \rho_0$ which proves the univalence of $D(F)$ in \mathbb{D}_{ρ_0} .

M	p	$\rho = \rho(M, p)$	$R = R(M, \rho(M, p))$	ρ'	R'
1.1296	2	0.0714741	0.0101601	0.0420157	0.00945379
2	2	0.0206783	0.00227639	0.0139439	0.00164502
2.2976	2	0.0155966	0.00151523	0.0106132	0.00108021
3	2	0.00922255	0.00067425	0.00626141	0.000482413
1.1296	3	0.071463	0.0101647	—	—
2	3	0.0206782	0.00227641	—	—
2.2976	3	0.0155966	0.00151523	—	—
3	3	0.00922254	0.000674251	—	—
1.1296	4	0.0714629	0.0101647	—	—
2	4	0.0206782	0.00227641	—	—
2.2976	4	0.0155966	0.00151523	—	—
3	4	0.00922254	0.000674251	—	—

TABLE 1. Values of ρ and R for Theorem 3 for $p = 2$, and the corresponding values of ρ' and R' of [7, Theorem 1.1] (for $p = 2$)

For any z with $|z| = \rho_0$, we have

$$\begin{aligned}
|H(z)| &= \left| \sum_{k=1}^p |z|^{2(k-1)} [z(G_{p-k+1})_z(z) - \bar{z}(G_{p-k+1})_{\bar{z}}(z)] \right| \\
&\geq \left| z(G_p)_z(0) - \bar{z}(G_p)_{\bar{z}}(0) \right| \\
&\quad - \left| z[(G_p)_z(z) - (G_p)_z(0)] - \bar{z}[(G_p)_{\bar{z}}(z) - (G_p)_{\bar{z}}(0)] \right| \\
&\quad - \left| \sum_{k=2}^p |z|^{2(k-1)} [z(G_{p-k+1})_z(z) - \bar{z}(G_{p-k+1})_{\bar{z}}(z)] \right| \\
&\geq \rho_0 \left[\lambda_0(M) - \sum_{k=2}^p \frac{T(M)\rho_0^{2(k-1)}}{(1-\rho_0)^2} - \frac{16M}{\pi^2} s_0 \arctan \rho_0 \right] \\
&= R
\end{aligned}$$

and the proof of the theorem is complete. \square

From Table 1, we see that Theorem 3 improves Theorem 1.1 of [7] for the case $p = 2$, and the results for the rest of the values of p are new. In Table 1, third and fourth columns refer to values obtained from Theorem 3 for cases $p = 2, 3, 4$ for certain choices of M , while the right two columns correspond to the values obtained from [7, Theorem 1.1] for the case $p = 2$.

Theorem 4. *Let $f(z) = |z|^{2(p-1)}G(z)$ be a p -harmonic mapping of \mathbb{D} satisfying $G(0) = J_G(0) - 1 = 0$ and $|G(z)| \leq M$, where $M \geq 1$ and G is harmonic. Then there is a constant ρ ($0 < \rho < 1$) such that $D(f)$ is univalent in \mathbb{D}_ρ , where ρ satisfies*

the following equation:

$$\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} = 0,$$

where the constants s_0 , $\lambda_0(M)$ and $T(M)$ are the same as in Theorem 3. Moreover, the range $D(f)(\mathbb{D}_\rho)$ contains a univalent disk \mathbb{D}_R , where

$$R = \rho^{2p-1} \left[\lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho \right].$$

Epecially, if $M = 1$, then $G(z) = z$, i.e. $f(z) = |z|^{2(p-1)}z$ which is univalent in \mathbb{D} .

Proof. Let $G(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}_n$. Using Lemmas B, C and Corollary 1, we have

$$|a_n| + |b_n| \leq T(M) \quad \text{for } n \geq 2.$$

Note that

$$J_G(0) = |a_1|^2 - |b_1|^2 = 1$$

and hence, by Lemmas A and B, we have

$$\lambda_G(0) \geq \lambda_0(M).$$

Next, we set $H = D(f) = |z|^{2(p-1)}D(G)$ and fix ρ with $\rho \in (0, 1)$. To prove the univalence of f , we choose two distinct points z_1, z_2 in \mathbb{D}_ρ . Let $\gamma = \{(z_2 - z_1)t + z_1 :$

$0 \leq t \leq 1\}$ and $z_2 - z_1 = |z_1 - z_2|e^{i\theta}$. Then

$$\begin{aligned}
|H(z_1) - H(z_2)| &= \left| \int_{[z_1, z_2]} H_z(z) dz + H_{\bar{z}}(z) d\bar{z} \right| \\
&= \left| \int_{[z_1, z_2]} p|z|^{2(p-1)} (G_z(z) dz - G_{\bar{z}}(z) d\bar{z}) \right. \\
&\quad + |z|^{2(p-1)} (z G_{z^2}(z) dz - \bar{z} G_{\bar{z}^2}(z) d\bar{z}) \\
&\quad \left. + (p-1)|z|^{2(p-2)} (z^2 G_z(z) d\bar{z} - \bar{z}^2 G_{\bar{z}}(z) dz) \right| \\
&\geq \left| \int_{[z_1, z_2]} [G_z(0)(p|z|^{2(p-1)} dz + (p-1)|z|^{2(p-2)} z^2 d\bar{z}) \right. \\
&\quad \left. - G_{\bar{z}}(0)(p|z|^{2(p-1)} d\bar{z} - (p-1)|z|^{2(p-2)} \bar{z}^2 dz)] \right| \\
&\quad - p \left| \int_{[z_1, z_2]} |z|^{2(p-1)} [(G_z(z) - G_z(0)) dz - (G_{\bar{z}}(z) - G_{\bar{z}}(0)) d\bar{z}] \right| \\
&\quad - \left| (p-1) \int_{[z_1, z_2]} |z|^{2(p-1)} \left[\frac{z}{\bar{z}} (G_z(z) - G_z(0)) d\bar{z} \right. \right. \\
&\quad \left. \left. - \frac{\bar{z}}{z} (G_{\bar{z}}(z) - G_{\bar{z}}(0)) dz \right] \right| \\
&\quad - \left| \int_{[z_1, z_2]} |z|^{2(p-1)} (z G_{z^2}(z) dz - \bar{z} G_{\bar{z}^2}(z) d\bar{z}) \right| \\
&\geq |z_1 - z_2| \left(\int_0^1 |z|^{2(p-1)} dt \right) \left\{ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho \right. \\
&\quad \left. - \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|) \rho^{n-1} \right\} \\
&> |z_1 - z_2| \left(\int_0^1 |z|^{2(p-1)} dt \right) \left[\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} \right].
\end{aligned}$$

Since there exists a unique ρ in $(0, 1)$ which satisfies the following equation:

$$\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} = 0,$$

we see that $H(z_1) \neq H(z_2)$ and so, $H(z)$ is univalent for $|z| < \rho_0$.

Furthermore, we observe that for any z with $|z| = \rho_0$,

$$\begin{aligned}
|H(z)| &= \rho_0^{2(p-1)} |z G_z(0) - \bar{z} G_{\bar{z}}(0) + z(G_z(z) - G_z(0)) - \bar{z}(G_{\bar{z}}(z) - G_{\bar{z}}(0))| \\
&\geq \rho_0^{2p-1} \left[\lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho_0 \right] \\
&= R.
\end{aligned}$$

The proof of the theorem is complete. \square

M	p	$\rho = \rho(M, p)$	$R = R(M, \rho(M, p))$	ρ'	R'
1.1296	2	0.0281673	0.0000106985	0.0194864	3.54498×10^{-6}
2	2	0.00856025	1.73218×10^{-7}	0.00623202	6.5415×10^{-8}
2.2976	2	0.00646284	6.4986×10^{-8}	0.0047235	2.47902×10^{-8}
3	2	0.0037942	1.00669×10^{-8}	0.00277162	3.83502×10^{-9}
1.1296	3	0.0281673	8.48819×10^{-9}	—	—
2	3	0.00856025	1.2693×10^{-11}	—	—
2.2976	3	0.00646284	2.71435×10^{-12}	—	—
3	3	0.0037942	1.44922×10^{-13}	—	—

TABLE 2. Values of ρ and R for Theorem 4 for $p = 2, 3$, and the corresponding values of ρ' and R' of [7, Theorem 1.2] (for $p = 2$)

We remark that Theorem 4 is an improved version of [7, Theorem 1.2] when $p = 2$. In order to be more explicit, we refer to Table 2 in which the third and fourth columns refer to values obtained from Theorem 4 for cases $p = 2, 3$ for certain choices of M , while the right two columns correspond to the values obtained from [7, Theorem 1.2] for the case $p = 2$.

5. THE REGION OF VARIABILITY

Definition 3. Let \mathcal{H}_p denote the set of all p -harmonic mappings of the unit disk \mathbb{D} with the normalization $f_{z^{p-1}}(0) = (p-1)!$ and $|f(z)| \leq 1$ for $|z| < 1$. Here we prescribe that $\mathcal{H}_0 = \emptyset$.

For a fixed point $z_0 \in \mathbb{D}$, let

$$V_p(z_0) = \{f(z_0) : f \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}\}.$$

Now, we have

Theorem 5. (a) If $p = 1$, then $V_1(z_0) = \{1\}$;
(b) If $p \geq 2$, $V_p(z_0) = \overline{\mathbb{D}}$.

Proof. We first prove (a). Let $f \in \mathcal{H}_1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$. By Parseval's identity and the hypotheses $|f(z)| \leq 1$ and $f(0) = 1$, we have

$$\begin{aligned} \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} (|h(re^{i\theta})|^2 + |g(re^{i\theta})|^2) d\theta \\ &= |a_0|^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \leq 1. \end{aligned}$$

This inequality implies that for any $n \geq 1$, $a_n = b_n = 0$ which gives that $f(z) \equiv 1$ for $z \in \mathbb{D}$. Thus, we have $V_1(z_0) = \{1\}$.

In order to prove (b), we consider the function

$$\phi(z) = \frac{z^{p-1} - w}{1 - w\bar{z}^{p-1}} = |z|^{2(p-1)} \sum_{n=1}^{\infty} w^n \bar{z}^{(n-1)(p-1)} + z^{p-1} - w - \sum_{n=1}^{\infty} w^{n+1} \bar{z}^{(p-1)n},$$

where $w \in \overline{\mathbb{D}}$ and $p \geq 2$.

Then $\phi_{z^{p-1}}(0) = (p-1)!$, $\Delta^p \phi = 0$ and therefore, $\phi \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}$. For each fixed $a \in \overline{\mathbb{D}}$, $z \mapsto f_a(z) = (z^{p-1} - a)/(1 - a\overline{z}^{p-1})$ is a p -harmonic mapping and $f_a(\mathbb{D}) \subset \mathbb{D}$.

Obviously, $a \mapsto f_a(z_0) = \frac{z_0^{p-1} - a}{1 - a\overline{z_0}^{p-1}}$ is a conformal automorphism of \mathbb{D} and the image of $\overline{\mathbb{D}}$ under $f_a(z_0)$ is $\overline{\mathbb{D}}$ itself. By hypotheses, we obtain that for any $g \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}$, $g(z_0) \in \overline{\mathbb{D}}$. Hence $V_0(z_0)$ coincides with $\overline{\mathbb{D}}$. The proof of this theorem is complete. \square

By the method of proof used in Theorem 5(a), we obtain the following generalization of Cartan's uniqueness theorem (see [5] or [27, p. 23]) for harmonic mappings.

Theorem 6. *Let f be a harmonic mapping in \mathbb{D} with $f(\mathbb{D}) \subseteq \mathbb{D}$ and $f_z(0) = 1$. Then $f(z) = z$ in \mathbb{D} .*

6. ESTIMATES FOR BLOCH NORM FOR BI- AND TRI-HARMONIC MAPPINGS

In the case of p -harmonic Bloch mappings, the authors in [10] obtained the following result.

Theorem 7. *Let f be a p -harmonic mapping in \mathbb{D} of the form (1.1) satisfying $B_f < \infty$, where*

$$B_f := \sup_{z, w \in \mathbb{D}, z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < \infty \quad \text{with} \quad \rho(z, w) = \frac{1}{2} \log \left(\frac{1 + \left| \frac{z-w}{1-\overline{z}w} \right|}{1 - \left| \frac{z-w}{1-\overline{z}w} \right|} \right).$$

Then

$$\begin{aligned} B_f &:= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left\{ \left| \sum_{k=1}^p |z|^{2(k-1)} (G_{p-k+1})_z(z) \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^p (k-1) \overline{z} |z|^{2(k-2)} G_{p-k+1}(z) \right| + \left| \sum_{k=1}^p |z|^{2(k-1)} (G_{p-k+1})_{\overline{z}}(z) \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^p (k-1) z |z|^{2(k-2)} G_{p-k+1}(z) \right| \right\} \\ (6.1) \quad &\geq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \left| \sum_{k=1}^p |z|^{2(k-1)} (G_{p-k+1})_z(z) \right| - \left| \sum_{k=1}^p |z|^{2(k-1)} (G_{p-k+1})_{\overline{z}}(z) \right| \right| \end{aligned}$$

and (6.1) is sharp. The equality sign in (6.1) occurs when f is analytic or anti-analytic.

Furthermore, if for each $k \in \{1, 2, \dots, p\}$, the harmonic functions G_{p-k+1} in (1.1) are such that $|G_{p-k+1}(z)| \leq M$, then

$$(6.2) \quad B_f \leq 2M\phi_p(y_0).$$

Here y_0 is the unique root in $(0, 1)$ of the equation $\phi'_p(y) = 0$, where

$$(6.3) \quad \phi_p(y) = \frac{2}{\pi} \sum_{k=1}^p y^{2(k-1)} + y(1 - y^2) \sum_{k=2}^p (k-1) y^{2(k-2)}.$$

The bound in (6.2) is sharp when $p = 1$, where M is a positive constant. The extremal functions are

$$f(z) = \frac{2M\alpha}{\pi} \operatorname{Im} \left(\log \frac{1+S(z)}{1-S(z)} \right),$$

where $|\alpha| = 1$ and $S(z)$ is a conformal automorphism of \mathbb{D} .

In order to emphasize the importance of this result, we recall that, when $p = 1$, (6.1) (resp. (6.2)) is a generalization of [12, Theorem 1] (resp. [12, Theorem 3]). In the case of $p = 2$ of Theorem 7, after some computation, one has the following simple formulation for biharmonic mappings.

Corollary 3. *Let $f = H + |z|^2 G$ be a biharmonic mapping of \mathbb{D} such that $B_f < \infty$. Then, we have*

$$(6.4) \quad B_f \geq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| |H_z + |z|^2 G_z| - |H_{\bar{z}} + |z|^2 G_{\bar{z}}| \right|$$

and

$$(6.5) \quad B_f \leq \frac{4M}{27\pi^3} \left(8 + 36\pi^2 + (4 + 3\pi^2)^{3/2} \right) \approx 30.7682M.$$

Proof. According to our notation, (6.1) is equivalent to (6.4). In order to prove (6.5), we first observe that (6.2) is equivalent to

$$B_f \leq 2M \sup_{0 < y < 1} \phi_2(y),$$

where

$$\phi_2(y) = \frac{2}{\pi} (1 + y^2) + y(1 - y^2).$$

Now, to find $\sup_{0 < y < 1} \phi_2(y)$, we compute the derivative

$$\phi_2'(y) = 1 + \frac{4}{\pi} y - 3y^2 = -3(y - y_0) \left(y - \frac{2 - \sqrt{4 + 3\pi^2}}{3\pi} \right)$$

so that $\phi_2'(y) \geq 0$ for $0 \leq y \leq y_0$ and $\phi_2'(y) \leq 0$ for $y_0 \leq y < 1$. Hence

$$y_0 = \frac{2 + \sqrt{4 + 3\pi^2}}{3\pi} \approx 0.82732$$

is the critical point of $\phi_2(y)$. Consequently, $\phi_2(y) \leq \phi_2(y_0)$. A simple calculation shows that

$$\begin{aligned} \phi_2(y_0) &= \frac{2}{\pi} (1 + y_0^2) + y_0(1 - y_0^2) \\ &= \frac{2}{\pi} \left(\frac{8 + 12\pi^2 + 4\sqrt{4 + 3\pi^2}}{9\pi^2} \right) + \left(\frac{2}{3\pi} + \frac{\sqrt{4 + 3\pi^2}}{3\pi} \right) \left(\frac{6\pi^2 - 8 - 4\sqrt{4 + 3\pi^2}}{9\pi^2} \right) \\ &= \frac{2}{27\pi^3} \left(16 + 42\pi^2 + 8\sqrt{4 + 3\pi^2} + \sqrt{4 + 3\pi^2} (3\pi^2 - 4 - 2\sqrt{3\pi^2 + 4}) \right) \\ &= \frac{2}{27\pi^3} \left(8 + 36\pi^2 + (4 + 3\pi^2)^{3/2} \right) \approx 15.3841 \end{aligned}$$

and therefore, $B_f \leq 2M\phi_2(y_0)$ which is the desired inequality (6.5). The result follows. \square

In the case of $p = 3$ of Theorem 7, we have

Corollary 4. *Let $f = H + |z|^2G + |z|^4K$ be a triharmonic (i.e. 3-harmonic) mapping of the unit disk \mathbb{D} such that $B_f < \infty$, where H , G and K are harmonic in \mathbb{D} . Then we have*

$$(6.6) \quad B_f \geq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| |H_z + |z|^2G_z + |z|^4K_z| - |H_{\bar{z}} + |z|^2G_{\bar{z}} + |z|^4K_{\bar{z}}| \right|$$

and

$$(6.7) \quad B_f \leq 2M\phi_3(y_1) \approx 4.037006M,$$

where $\phi_3(y_1) = \sup_{0 < y < 1} \phi_3(y)$ and

$$\phi_3(y) = \frac{2}{\pi}(1 + y^2 + y^4) + y(1 + y^2 - 2y^4).$$

Proof. Set $p = 3$ in Theorem 7. Then, (6.6) is equivalent to (6.1) and therefore, it suffices to prove (6.7). The choice $p = 3$ in (6.2) shows that

$$B_f \leq 2M \sup_{0 < y < 1} \phi_3(y),$$

where $\phi_3(y)$ is obtained from (6.3).

We see that $\phi_3(y)$ has a unique positive root in $(0, 1)$. Also,

$$\phi'_3(y) = \frac{4}{\pi}(y + 2y^3) + 1 + 3y^2 - 10y^4.$$

Computations show that $\phi'_3(y) \geq 0$ for $0 \leq y \leq y_1$ and $\phi'_3(y) \leq 0$ for $y_1 \leq y < 1$. Hence

$$y_1 \approx 0.891951$$

is the only critical point of $\phi_3(y)$ in the interval $(0, 1)$. It follows that

$$\phi_3(y) \leq \phi_3(y_1) \approx 2.018503.$$

Thus, $B_f \leq 2M\phi_3(y_1)$ which is the desired inequality (6.7). \square

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